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Types:

- Heat equation $\quad T_t = c^2 T_{xx}$
- Wave Equation $\quad u_{tt} = c^2 u_{xx}$
- Laplace Equation $\quad u_{xx} + u_{yy} = 0$

General Question Solving

1. Find the bounds (BCs and ICs)
2. Separate the general equation into dimensions: $T(x, y) = F(x) \cdot G(x)$
3. Solve the equation using the BCs & ICs.
4. Write the full solution

Interpolation

Lagrange polynomial, fitting a polynomial between points:

$$P_1(x) = L_0(x) f_0 + L_1(x) f_1$$

Interpolates between points f_0 and f_1. However, they can easily blow out to ∞ when higher order.

Splines, offer another method - more accurate & coupling of equations, having shared gradients & not discontinuous.

Stencils

$$u(x + \Delta x) = u(x) + \sum_{n=1}^{\infty} \frac{\Delta x^n}{n!} \frac{\delta^n u}{\delta x^n}(x)$$

Taylor Series based approximation, with $\frac{\delta u}{\delta x}|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$

When calculating for a value i.e., $f'(x)$ or $f''(x)$, we want the value such that:

$$\begin{aligned} c_0 + c_1 + c_2 &= 0 \\ c_0(x_i - x_0) + c_1(x_{i+1} - x_0) + c_2(x_{i+2} - x_0) &= 1 \\ c_0(x_i - x_0)^2 + c_1(x_{i+1} - x_0)^2 + c_2(x_{i+2} - x_0)^2 &= 0 \end{aligned}$$

These are simply found by combining the Taylor series, $f(x_0) = x_i$, $f(x_0 + h) = x_i + hx_i' + hx_i'' + \ldots$

to find the certain weights, use the subs $x_i - x_0 = x_i, x_{i+1} - x_0 = x_i + h, \ldots$

Which also fields an first order error $\mathcal{O}(h)$ as dependent on $f''(x_i)$.

Fourier Series Approximation

- Even function, periodic $\longrightarrow$ Fourier Cosine Series
- Odd function, periodic $\longrightarrow$ Fourier Sine Series
- *if periodic but neither* $\longrightarrow$ Fourier Series with both sin & cos terms
- Even function, 'singular' $\longrightarrow$ Fourier Cosine Integral
- Odd function, 'singular' $\longrightarrow$ Fourier Sine Integral

Solving PDE Systems

Conditions:

$c_2^2 < 4c_1c_3$	elliptic	Laplace
$c_2^2 = 4c_1c_3$	parabolic	heat
$c_2^2 > 4c_1c_3$	hyperbolic	wave

Dirichlet boundaries - exact value boundaries

Neumann boundaries - gradient based boundaries.

Use the general equation ($T(x, y) = F(x)G(y)$), and the only 'interesting'/non-trivial behaviour is at the negative value,

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = -p^2$$

Solving Heat Equation **Analytically**

When solving analytical solutions, first use the general equation: $T(x, y) = F(x)G(x)$, and then split into temporal & spatial components*:

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = -p^2 \Rightarrow \begin{cases} G_t + p^2 c^2 G = 0 \\ F_{xx} + p^2 F \quad = 0 \end{cases}$$

*To split use $T_t = F(x)G'(t)$ and $T_{xx} = F''(x)G(t)$, then congregate the like terms

Then use the BCs to solve the spatial solution, use ODE solution $\phi_{xx} + p^2\phi = 0 \Rightarrow \phi(x) = a\cos(px) + b\sin(px)$, which will usually give only a sine solution (due to the 0 BC at $x = 0$), i.e., $b\sin\left(\frac{2\pi n}{L}\right)$ with $n \in 0, 1, 2\ldots$

While the temporal component can be simplified similarly, $\phi_t + c^2p^2\phi \Rightarrow \phi(t) = Ae^{-c^2p^2t}$

These are then combined into the full solution (with the coefficients A and b combined), $\sum_{n=1}^{\infty} B_n^* \sin\left(\frac{2\pi n}{L}\right) e^{-\left(\frac{2\pi}{L}\right)^2 t}$. Then the IC can be used to solve for the B_n^*, if the IC contains sine terms it can be directly correlated to give the entire analytical solution (no summation).

To find B_n^* normally use the IC condition as $f(x)$ and then sub into $B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,\mathrm{d}x$

Inhomogeneous HEs

Use this equation $u(x, t) = u_{ss}(x) + w(x, t)$

1. Need to transform into a steady state solution, T_{ss} with $T_{\text{ss}} = Ax + B$

"Right boundary"		"Left boundary"		T_{ss}
Dirichlet	Neumann	Dirichlet	Neumann	
α		β		$\alpha + (\beta - \alpha)\frac{x}{L}$
	α	β		$\alpha(x - L) + \beta$
α			β	$\alpha + \beta x$
	α		β	$\alpha x + (\beta - \alpha)\frac{x^2}{2L}$

2. Now we use the fact that $T(x, t) = T_H(x, t) + T_{\text{ss}}(x) \Rightarrow T_H(x, t) = T(x, t) - T_{\text{ss}}(x)$, as it is homogenous it now becomes $T_x^H(0, t) = T_x^H(L, t) = 0$. While the IC becomes $T^H(x, 0) = T_x^0(L - x)$

   Using the known Eigenfunction:

   $$T_n^H(x) = A_n\cos(px) + B_n\sin(px)$$

   Which leads to $T^H(x, 0)\sum_{n=1,\infty} A_n\cos(px)$
3. Need to find the values of the coefficients, A_n:

   $$A_n = \frac{2}{L}\int_0^L f(x)\cos(px)\,\mathrm{d}x$$

Finding Dominance Lengths

To find the length of time until a mode in the HE is dominant by a certain amount, use $(\lambda_n)^2 = \frac{c^2n^2\pi^2}{L^2}$ then use $\frac{e^{-(\lambda_2)^2 t}}{e^{-(\lambda_1)^2 t}} \sim 10^{-6}$

Numerically

Order of accuracy, where the error is terms removed from the approximation – the truncation error.

Numerical solutions should be:

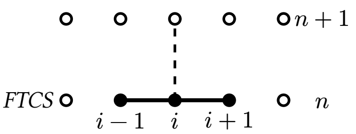
1. Stable
2. Consistent
3. Verified & Validated

Error norms $\mathcal{L}^n$, where $\mathcal{L}^n = \left(\Delta x \sum_{i=1}^{nx} |f_i - f(x_i)|^p\right)^{1/p}$, so $\mathcal{L}^1$ is mean absolute error ($\frac{\sum|y_i - \hat{y}_i|}{n}$), $\mathcal{L}^2$ is the RMS ($\sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n}}$) and $\mathcal{L}^\infty$ is the maximum error.

Then $\mathcal{O}(\mathcal{L}^n)$ is the convergence rate of the errors which should converge to the error term.

FTCS

Forward-in Time, Cental-in Space



$$T_i^{n+1} = \sigma T_{i-1}^n + (1 - 2\sigma)T_i^n + \sigma T_{i+1}^n$$

von Neumann Stability

With the FTCS scheme, solve it in an exponential form,

$$T(x,t) = \sum_{m=-\infty}^{\infty} e^{B_m n\Delta t} e^{Ikmi\Delta x}$$
$$\Rightarrow \underbrace{G^n}_{\text{time exp. decay}} \times \underbrace{e^{Ikmi\Delta x}}_{\text{space fourier series}}$$

Then analyse G^n and see its conditions; in FTCS case – $\Delta t \leq \frac{\Delta x^2}{2\alpha}$ with $\alpha = c$ mostly. Usually use quadratic formula for this.

Creating 'Schemes'

To create a CTCS or a BTCS etc schema, can use these approximations:

$$q_{xx} \approx \frac{q_i^{n+1} - 2q_i^n + q_i^{n-1}}{(\Delta t)^2} \qquad q_x \approx \frac{q_i^{n+1} + q_i^{n-1}}{2\Delta t} \qquad q_x \approx \frac{q_i^{n+1} + q_i^n}{\Delta t}$$

Order of Accuracy

To quantify the order of accuracy,

1. Use the Taylor series expansion for the point, $u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t}\big|_i^n + \frac{(\Delta t)^2}{2}\frac{\partial u}{\partial t}\big|_i^n + \frac{(\Delta t)^3}{6}\frac{\partial u}{\partial t}\big|_i^n + \mathcal{O}(4)$
2. Then substitute into the scheme, where terms will usually cancel (i.e., $u_i^{n-1} = u_i^n - \Delta t \frac{\partial u}{\partial t}\big|_i^n + \ldots$)
3. Rearrange to the original scheme, i.e., $\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = \frac{\partial u}{\partial t}\big|_i^n$

Unfinished

finish pls.....

BTCS

Backward-in time, central-in space — 'implicit' scheme,

$$T_i^n = (1+2\sigma)T_i^{n+1} - \sigma T_{i+1}^{n+1} - \sigma T_{i-1}^{n+1}$$

This scheme uses terms in the 'future', so is usually solved *simultaneously* with matrices. This scheme is also unconditionally stable.

Wave Equation

This is a hyperbolic equation which simulates the propagation of a wave in a medium.

Assumptions:

1. Constant ρ, same density of material and of wave & perfectly elastic
2. $T \gg mg$ — gravity is negligible
3. Small u/θ — angles in the wave are small, and can be subject to the small angle approximation:
   - $\sin\theta = \tan\theta = \theta$

$\lambda_n = \frac{c\pi}{L}$

Analytical Solution

1. BCs: $u(0,t) = 0$, Neumann *(describes the gradient at the endpoint)* $u_{x(L,t)} = 0$. IC $u_t(x,0) = 0$ & $u(x,0) = f(x)$
2. Using $F_{xx} + p^2F = 0$ and $G_{tt} + c^2p^2G = 0$
3. Use the PDE for F side, leading to $F(x) = a\cos(px) + b\sin(px)$ which is simplified using the BCs.
4. Same with the G side, except $\lambda_n = cp$, and need to find G_n' which is simplified with the BCs again.
5. Combine the coefficients into B_n^* which can be simplified using the initial displacement, $f(x)$

$\Longrightarrow f_n = \frac{\lambda_n}{2\pi}, T_n = \frac{2\pi}{\lambda_n}$ with $p = \frac{n\pi}{2L}$

CTCS — Numerical

Central-in time, central-in space scheme,

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{\Delta t^2 c^2}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

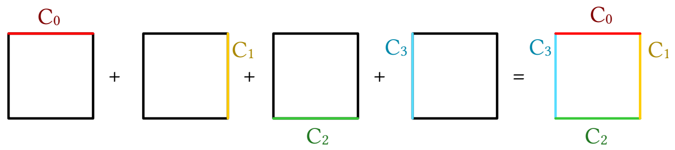
This is also an explicit scheme, stable when $\Delta t \leq \frac{\Delta x}{c}$

As this scheme is dependent on u^{n-1}, which could equate to u^{-1}, so this creates a *starting problem* which can be solved with:

- Assume that $\frac{du}{dt} = g(x)$ and then use central difference approximation: $u_{t,i}^0 \simeq \frac{u_i^1 - u_i^{-1}}{2\Delta t} = g$

Laplace/Poisson Equations

Find the steady state behaviour of an equation, i.e., heat equation (2D *HE*: $T_t = c^2T_{xx} + c^2T_{yy}$) This has boundary conditions in 2D! So they need to be super-*imposed* upon one another.



Finding the boundary conditions, use normal PDE methods with $\frac{F''}{F} = -\frac{G''}{G} = -p^2$

To move the axes for C_0, just swap y with $y - b$ as have shifted up by b units. To 'rotate' axes like for C_3 can simply swap x with y.

The answer is just the summation of all the individual ODE solutions

CTCS — Numerical

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right]$$

Gauss-Jacobi Method

Use a modified CTCS scheme to better allow for matrix calculation — defined at discrete timesteps,

$$u_{i,j}^{k+1} = \frac{1}{4}\left[u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k\right]$$

∞ & semi-∞ domains

Need to use the Laplace transform to 'transition' the problem into a more solvable one, while can use the Fourier Integral for non-periodic functions.

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,\mathrm{d}v\right]\mathrm{d}w$$

Then can use the error function to simplify $\mathrm{erf} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$

Fourier Transforms

Converts a complex frequency into a product of a series of fundamental frequencies. Goes from spatial to frequency domain.

Nyquist frequency: $\frac{f_{\text{samp}}}{2}$

To make:

1. Take the Laplace transform in t, yielding: $U_{xx} - \frac{s^2}{c^2}U = 0$
2. Use the general soln. to the ODE, $U(x,s) = A(s)e^{-\frac{sx}{c}} + B(s)e^{-\frac{sx}{c}}$
3. Apply the BCs to yield $U(x,s) = F(s)e^{-\frac{sx}{c}}$
4. Use second shift theorem to turn back into the time-domain, $25\cos\left(\frac{2\pi(t-\frac{x}{c})}{75}\right)e^{-0.05(t-\frac{x}{c})}\mathscr{H}(t - \frac{x}{c})$

Forcing Term

A forcing term converts the original equation into $U_{xx} - \frac{s^2}{c^2}U = -\frac{Q(s)}{c^2}$, with $Q(s) = \mathcal{L}(q(t))$

Laplace Transformations

Turn a function into a more 'manageable' form,

$$F(s) = \mathcal{L}\{f\}$$
$$= \int_0^\infty e^{-st}f(t)\,\mathrm{d}t$$

with s being a new Laplace complex variable.

Common transformations:

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0)$
$\int f(v)\,\mathrm{d}v$	$\frac{F(s)}{s}$

Second shift theorem, $\mathcal{L}(f(t-a)\mathcal{H}(t-a)) = e^{-as}F(s)$, with $\mathcal{H}$ being the Heaviside *(step)* function: $\mathcal{H}(x) = \begin{cases} 0 \text{ if } x<0 \\ 1 \text{ if } x \geq 0 \end{cases}$

Summaries

Domain / BCs	Governing Equation		
	Heat / Diffusion	Wave	Laplace
Finite domain Dirichlet / Neumann	Fourier series		
∞ domain	Fourier integral *(Fourier transform)*	–	–
Semi-∞ domain + forcing/time varying BC	Laplace transform		–

Integrals

with $+c$ **on all**

$$\int x\cos(ax)\,\mathrm{d}x = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax$$

$$\int x\sin(ax)\,\mathrm{d}x = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$$

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

$$\int u(x)v'(x)\,\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\,\mathrm{d}x$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)|$$