Amme2500 Dynamics

Kinematics & Kinetics

$a = \frac{dv}{dt}$ $\quad \int_{v_0}^{v} dv = \int_0^t a\,dt$

$v = \frac{ds}{dt}$ $\quad \int_{s_0}^{s} ds = \int_0^t v\,dt$

$v\,dv = a\,ds$ $\quad \int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a\,ds$

Where $\int_{v_0}^{v} \Rightarrow v - v_0 = \Delta v$

Normal n and tangential t coordinates

$\mathbf{a} = \frac{dv}{dt}\mathbf{u}_t + \left(\frac{v^2}{\rho}\right)\mathbf{u}_n$

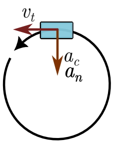
$xy \to nt$:

$a_t = \frac{\dot{x}a_x + \dot{y}a_y}{v}$, $a_n = \frac{|\dot{x}a_y - \dot{y}a_x|}{v}$

Radius of curvature at a point: $\rho = \left|\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\right|$

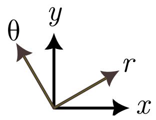
Centripetal accel. $a_c = \frac{v^2}{r} = \omega^2 r$, centripetal force $F_c = \frac{mv^2}{r} = m\omega^2 r$

Circular motion: $a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta}$, $a_t = \dot{v} = r\ddot{\theta}$



$\frac{1}{a_x} dv_x = dt$

Polar coordinates $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{x}{y}\right)$



To convert between, $\dot{v} \cdot \hat{\mathbf{e}}_r$, as the dot product finds the scalar quantity along the vector.

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta$$

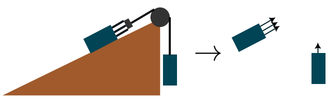
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_r + \left(r\ddot{\theta}_2\dot{r}\dot{\theta}\right)\hat{\mathbf{e}}_\theta$$

$\hat{\mathbf{e}}_r$ is the unit vector in r direction

$r_{AB} = r_{A/B} = r_A - r_B$, A with *respect* to B

For connected particle systems, use the constraint of a limited length of the connector (string), $5s_A + l_{CD} + s_B = l_\Sigma \implies 5v_A + v_B = 0 \implies 5a_A + a_B = 0$

Use constraint based solving when forces



Turns into,

$$\swarrow +ve \quad -3T = m\ddot{x} \; ①$$
$$\downarrow +ve \quad -T = m\ddot{x} \; ②$$
$$3\ddot{x}_1 + \ddot{x}_2 = 0 \quad ③$$

For drag based problems — use terminal velocity, where $\ddot{x} = 0$, $\therefore mg = kv_t$

Kinetics

Work & energy

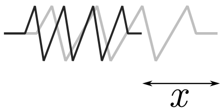
Kinetic $\boldsymbol{T}$, work $\boldsymbol{U}$, potential $\boldsymbol{V}$

With work being, $U = Fd$

$$T_2 = \overbrace{\sum U_{1-2}}^{\text{work done}} + T_1$$

$\Delta(T + V) = U$

Force	$U_{1-2} = \int_{r_1}^{r_2} F\,dr$
Gravity	$V = mgy$
Spring	$V = \frac{1}{2}ks^2$
Torsional Spr.	$V = \frac{1}{2}k_T\theta^2$ $\quad \theta$ in rad
Linear Motion	$T = \frac{1}{2}mv^2$
Rotational	$T = \frac{1}{2}I\omega^2$



Springs in parallel: $k_{eq} = k_1 + k_2 + \ldots$ & $x_{total} = x_1 = x_2 = \ldots$

In series: $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$ & $x_{total} = x_1 + x_2 + \ldots$

$$T_2 + V_2 = \sum U_{1-2} + T_1 + V_1$$

This is different as the $\sum U_{1-2}$ is all work done by **non**-conservative forces — friction (i.e., $\int_{s_1}^{s_2} \mu N\,ds$), air resistance. And the conservative forces (gravity, spring) transferred to V.

When two objects collide,

$$\text{coefficient of resitution: } e = \frac{v_{B2} - v_{A2}}{v_{A1} - v_{B1}}$$

this is in the normal n plane and v is magn.

Then the tangential momentums (mv) are equal.

Efficiency: $\varepsilon = \frac{P_{out}}{P_{in}} = \frac{U_{out}}{U_{in}}$

Angular Impulse & Momentum

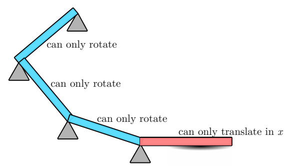
Momentum $G = m\mathbf{v}$

$$H_{O2} = \int_{t1}^{t2} \sum M_0\,dt + H_{O1}$$

$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}_{net}\,dt = \Delta\mathbf{G}$

Degrees of Freedom (DoF), minimum amount of coordinates (translations x, y, z + rotations $\theta_x, \theta_y, \theta_z$) to completely describe the system.

Can intuit the DoF, each free body in a system needs its x, y, θ; then see how the *freedom* is 'taken away' due to the connections.



DoF $= \sum$ Free DOF $-$ Constraints $= 12 - 10 = 2$

Vibration

Spring force: $F = -kx$, dampener force: $F = -b\dot{x}$

Free Vibration — Undamped

$$\ddot{x} + \frac{k}{m}x = \ddot{x} + \omega_n^2 x = 0$$

With $\omega_n = \sqrt{\frac{k}{m}}$, k being spring const.

Turned into $x = C\sin(w_n t + \phi)$ Period: $\tau = \frac{2\pi}{\omega_n}$, freq.: $f = \frac{\omega_n}{2\pi}$

Can **rebase** a system to static equilibrium point where $F_s = F_G \Leftrightarrow kd = mg$, use for calcs.

$X = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$ from general eq: $x(t) = X\cos(\omega_n t - \phi)$

Forced Vibration — Undamped

$$\ddot{x} + \frac{k}{m}x = \ddot{x} + \omega_n^2 = \frac{F_0}{m}\sin(\omega_0 t)$$

With $\boldsymbol{\omega}_0$ being **freq.** of *driving force*

Amplitude: $X = \frac{F_0/_k}{1 - \left(\frac{w_0}{w_n}\right)^2}$

Magnification factor: $\text{MF} = \frac{X}{F_0/_k} = \frac{1}{1 - \left(\frac{w_0}{w_n}\right)^2}$

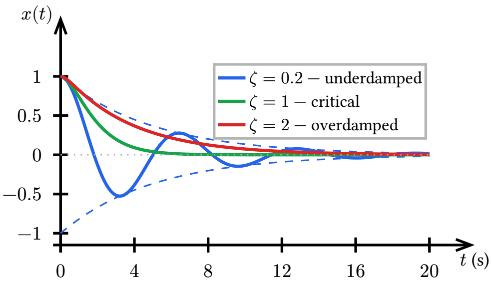
Free Vibration — Damped

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Dampening ratio: $\zeta = \frac{b}{2m\omega_n}$

Types of systems:

1. Overdamped, $\zeta > 1$
2. Critically damped, $\zeta = 1$
3. Underdamped, $\zeta < 1$ – only system to oscilate when decaying



Forced Vibration — Damped

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin(\omega_0 t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f\sin(\omega_0 t + \Phi)$$

$$\text{Amplitude: } X = \frac{f}{\omega_n^2\sqrt{\left(1 - \frac{w_0^2}{w_n^2}\right)^2 + \left(2\zeta\frac{w_0}{w_n}\right)^2}}$$

$$\text{Phase Angle: } \phi = \Phi - \arctan\left(\frac{2\zeta\frac{\omega_0}{\omega_n}}{1 - \frac{\omega_0^2}{\omega_n^2}}\right)$$

Rigid Body

$\frac{d\theta}{dt} = \dot{\theta} = \omega$

$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \dot{\omega} = \alpha$

$\alpha\,d\theta = \omega\,d\omega \implies \ddot{\theta}\,d\theta = \dot{\theta}\,d\dot{\theta}$

$v = r\dot{\theta} = r\omega$

$\mathbf{v} = \dot{\mathbf{r}} = \omega \times \mathbf{r}$

Normal

$a_n = r\dot{\theta}^2 = r\omega = \frac{v^2}{r}$

$\mathbf{a}_n = \omega \times (\omega \times \mathbf{r}) = -\omega^2\mathbf{r}$

Tangential

$a_t = r\ddot{\omega} = r\alpha$

$\mathbf{a}_t = \alpha \times \mathbf{r}$

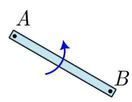
Relative Motion Analysis

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_A + \mathbf{v}_{B/A} \\ &= \mathbf{v}_A + \omega \times \mathbf{r}_{B/A}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{A}_{B/A} \\ &= \mathbf{a}_A - \underbrace{\omega^2\mathbf{r}_{B/A}}_{\text{normal}} + \underbrace{\boldsymbol{\alpha} \times \mathbf{r}_{B/A}}_{\text{tangential}}\end{aligned}$$

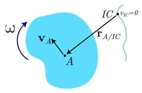
Steps:

1. $v = r\omega$ from known
2. Then $\mathbf{v}_B = \mathbf{v}_A + \ldots$, if ω unknown usually can use $v = r\omega$
3. Use known in $\mathbf{a}_C = \mathbf{a}_B + \ldots$ and solve for ans.

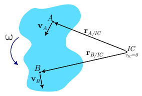


$r_{B/A} \Rightarrow \mathbf{A} \rightarrowtail \mathbf{B} \approx 0.4\mathbf{i} - 0.2\mathbf{j}$

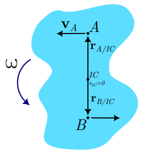
Instantaneous centre of mass/zero-$\vec{v}$, draw lines from velocities to work out intersection point of the `IC`



Case 1: v_A and ω are known
1. Construct a line segment $\perp$ to v_A @ A
2. Distance is $r_{\frac{A}{\mathtt{IC}}} = \frac{|v_A|}{\omega}$



Case 2: two **non**-parallel velocities
1. Construct $\perp$ lines from A & B
2. Intersection point is `IC`, $r_{A/\mathtt{IC}} = \frac{r_{B/A}\cdot\hat{\mathbf{n}}_{B\perp}}{\hat{\mathbf{n}}_A\cdot\hat{\mathbf{n}}_{B\perp}}$



Case 3: two parallel vectors
1. Use proportional triangles ⊿ to find `IC`
2. Distance is $\frac{|v_A|}{r_A/\mathtt{IC}} = \frac{|v_B|}{r_B/\mathtt{IC}} = \omega$

The `IC` is used with $\mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{r}_{B/A}$ as $\mathbf{v}_A = 0$ so can just use the $\omega \times \mathbf{r}_{B/\mathtt{IC}}$

☆ Use `IC` when undergoing *general planar motion*, i.e., both rotation and <u>and</u> translation, otherwise can just use $v = \omega r$.

Conservation of angular momentum $I\omega = I'\omega$, and need to use CoM for potential energy.

The sum of forces is equal to the equivalent accel. at the centre,

$$\sum_{i=1}^{n} F_i = ma_G$$

$$\sum M_G = I_G\alpha$$

$$\sum M_P = \mathbf{I}_G\alpha + \mathbf{r}_{G/P} \times m\mathbf{a}_G$$

rectilinear motion

Mass moment of inertia

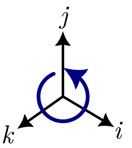
Parallel axis theorem — can 'move' the inertia, $I = I_G + mr^2$

$$\text{Kinetic, } T = \overbrace{\tfrac{1}{2}mv_G^2}^{\text{translational}} + \tfrac{1}{2}I_G\omega^2$$

$$T_2 = \sum U_{1-2} + T_1$$

$U_{1-2} = \int_{\theta_1}^{\theta_2} M \,\mathrm{d}\theta = M(\theta_2 - \theta_1)$, M is the moment

Maths



$\therefore \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$ ...

Degrees to radians, $° \times \frac{\pi}{180}$

$\mathrm{km\,h^{-1}}$ to $\mathrm{m\,s^{-1}}$, $\mathrm{km\,h^{-1}} \div 3.6$

Homogenous differential equations,

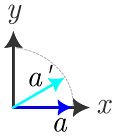
Roots	Soln.
Distinct $\zeta > 1$	$Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
Equal $\zeta = 1$	$(A + Bx)e^{-\lambda x}$
Complex $\zeta < 1$	$e^{-\alpha x}(A\cos(\beta x) + B\sin(\beta x)) \sim r = \alpha \pm \beta i$

Small angle approximation, $\sin\theta \approx \tan\theta \approx \theta$

Deriv ↥, $\dot{\text{d}}$ot. Integral ↧, remove.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c}) \times \mathbf{b} - (\mathbf{a}\cdot\mathbf{b}) \times \mathbf{c}$$

Rotation matrix: $R_{\circlearrowleft}(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$, rotating $\begin{pmatrix}1\\0\end{pmatrix}$ by 30°, $\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \times \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$



$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Conventions: $\circlearrowleft +ve$; $\uparrow, \rightarrow +ve$

Derivatives,

$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)g(t)]$	$f'(t)g(t) + f(t)g'(t)$
$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{f(t)}{g(t)}\right]$	$\frac{f'(t)g(t) - f(t)g'(t)}{[g(t)]^2}$
$\frac{\mathrm{d}}{\mathrm{d}t}[f(g(t))]$	$f'(g(t)) \times g'(t)$

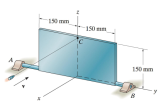
Process

1. Draw FBD!!
2. Look if fits into energy equations
3. Generalised equation

Examples

Example R7

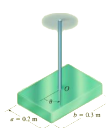
A 15 kg plate is free to rotate about the y -axis owing to the bearing supports at A and B. When the plate is balanced in the vertical plane, a 3 g bullet is fired into the plate, becoming embedded within it, with a velocity $2000\,\mathrm{m\,s^{-1}}$ as shown. Compute the *angular velocity* of the plate at the instant it has rotated 180°.



1. Use the conservation of angular momentum, $\sum \mathbf{H}_{AB,1} = \sum \mathbf{H}_{AB,2}$, so can use $\mathbf{H}_{AB,1} = \mathbf{r} \times m\mathbf{v}_{b,1}$.
2. Then equate $\mathbf{H}_{AB,2}$ with its $I_{xx}\omega_x + ...$ components where the only direction of rotation is in ω_y, so $\omega_x = \omega_z = 0$. Use $I_{yy} = \frac{1}{12}ml^2 + md^2$, parallel axis theorem, to solve.
3. Use energy formula $T_3 + V_3 = \sum U_{2-3} + T_2 + V_2$, where T_2 & $T_3 = \frac{1}{2}I_{yy}\omega_{y,3}^2$ and V_2 & $V_3 = mgh$ to solve for $\omega_{y,3}$.

Example R3

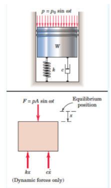
The 10 kg rectangular plate shown is suspended at its centre from a rod having a torsional stiffness k = $1.5\,\mathrm{N\,m\,rad^{-1}}$. Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



1. Equation of motion: $\sum M_0 = I_0\alpha \Rightarrow -k\theta = I_0\ddot{\theta}, \therefore \ddot{\theta} + \frac{k}{I_0}\theta = 0$
2. Use general from, $\ddot{\theta} + \omega_n^2\theta = 0$, so $\omega_n = \sqrt{\frac{k}{I_0}}$
3. Calculate I_0 with the *rod about centre* in `2D` so $I_0 = \frac{1}{12}M(l^2 + h^2)$

Example R4

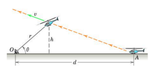
The 40 kg piston is supported by a spring of modulus k = $35.6\,\mathrm{kN\,m^{-1}}$. A dashpot of damping coefficient b = $1180\,\mathrm{N\,s\,m^{-1}}$ acts in parallel with the spring. A fluctuating pressure $P = 4.3\sin 30t$ [Pa] acts on the piston, whose top surface area is $0.05\,\mathrm{m}^2$. Determine the steady-state displacement as a function of time and the maximum force transmitted to the base.



1. Use equation of motion, $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin(\omega_0 t)$
2. Use $\omega_n = \sqrt{\frac{k}{m}}$, recognise that system is underdamped as $\zeta < 1$
3. Solve for X' with $\frac{F_0/k}{\sqrt{\left(1-\frac{w_0^2}{w_n^2}\right)^2 + \left(2\zeta\frac{w_0}{w_n}\right)^2}}$
4. Solve for ϕ' with $\tan^{-1}\left[\frac{2\zeta\frac{\omega_0}{\omega_n}}{1-\left(\frac{\omega_0^2}{\omega_n^2}\right)}\right]$
5. Use X_p equation with vals, $X_p = X'\sin(\omega_0 t - \phi')$
6. Then the force transmitted is the sum of spring & dampener, $\therefore F_b = kX_p + b\dot{X}_p$. So find $F_{b,\max}$

`MUGS` Q1.2

A helicopter starts from rest at a point A and travels along the straight line path with a constant acceleration a. At this instant the angle of inclination of the tracking device is θ and the distance from O to A is d. Neglect the height of the tracking device off the ground.



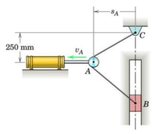
If the speed is v when the altitude of the helicopter is h, for the tracking device at O, determine $\dot{r}$ and $\dot{\theta}$; and $\ddot{r}$ and $\ddot{\theta}$.

1. Use the geometry to solve for r, the distance from the sensor to helicopter, $r = \frac{h}{\sin\theta}$
2. Solve for the velocity in x, y, $\mathbf{v} = \frac{v}{s}\begin{bmatrix}r\cos\theta - d \\ h\end{bmatrix}$
3. Convert to polar r, θ coordinates, using $\dot{r} = \dot{v}\cdot\hat{\mathbf{e}}_r$, so $\frac{v}{s}\begin{bmatrix}(r\cos\theta - d) \\ h\end{bmatrix} \cdot \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$. Similarly solve for θ coordinate with $\dot{v}\cdot\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\theta = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$
4. This gives $\mathbf{v}\cdot\hat{\mathbf{e}}_r = \dot{r} = \frac{v}{s}(r - d\cos\theta)$, and $\dot{\theta} = v\frac{d\sin\theta}{rs}$

I. For the acceleration, use $\mathbf{v}\,\mathrm{d}v = \mathbf{a}\,\mathrm{d}s$, then integrate to $a = \frac{v^2}{2s}$
II. Need to write in polar, $\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\mathbf{e}}_\theta$
III. Use dot products to solve for a_r, a_θ with $\mathbf{a} = \frac{a}{s}\begin{bmatrix}r\cos\theta - d \\ h\end{bmatrix}$

2024 Q3

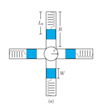
The rod of the fixed hydraulic cylinder is moving horizontally to the left with a constant speed $v_A = 35\,\mathrm{mm\,s^{-1}}$. The cord acts on a slider B which is constrained to move only vertically. The total length of the cord connecting B and C through the pulley A is 1050 mm, and the effects of the radius of the small pulley A may be neglected. What is the velocity of slider B at the instant when $s_A =$ 300 mm?



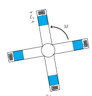
1. Use geometric pulley constraints, $L_{AC} + L_{AB} = 1050$
2. Find L_{AB} with geometry
3. Using the length equation $L = L_{AC} + L_{AB} = \sqrt{s_A^2 + 250^2} + \sqrt{s_A^2 + (y_B - 250)^2}$ differentiate to $0 = \frac{s_A\cdot\dot{s}_A}{\sqrt{s_A^2 + 250^2}} + \frac{s_A\cdot\dot{s}_A + (y_B - 250)\cdot\dot{y}_B}{\sqrt{s_A^2 + (y_B - 250)^2}}$

`MUGS` Q3.3

A centrifuge, with moment of inertia about its central axis I, contains 4 square blocks of mass m which are free to slide along radially pointing paths. Springs support the blocks and stop them from slamming into the walls of the device.



A moment of M is applied to the device for one rotation of the device. The centrifuge therefore begins to spin, causing the springs to compress. The moment of inertia of a square object about its centre is $I_C = \frac{1}{6}mW^2$.



1. Use energy equation, $T_1 + V_1 + \sum U_{1-2} = T_2 + V_2$ where $\sum U_{1-2} = \int_{s_1}^{s_2} F\,\mathrm{d}s + \int_{\theta_1}^{\theta_2} M\,\mathrm{d}\theta$
2. Solve for T_2, with $\frac{1}{2}I_0\omega^2$ from the centrifuge structure and $4 \times \frac{1}{2}I_B\omega^2$ for the weights.
3. Then use V_2 is the spring force $4 \times \frac{1}{2}kx^2$
4. Solve for ω using centripetal force which balances the spring force $F = kx = \frac{mv^2}{r} \Rightarrow kx = m\omega^2 r$, and sub ω into energy eqn.

Symbol	Description	Units	Derivation
ω	Radial velocity	$\mathrm{rad\,s^{-1}}$	
α	Radial acceleration	$\mathrm{rad\,s^{-2}}$	
ω_n, ω_0	Frequency	Hz	
v	Velocity	$\mathrm{m\,s^{-1}}$	
a	Acceleration	$\mathrm{m\,s^{-2}}$	
m	Mass	kg	
F	Force	N	
U, K, T, W	Energy/Work	J	$\int \mathbf{F}\,\mathrm{d}s = \int \mathbf{F}v\,\mathrm{d}t$
p	Momentum	$\mathrm{kg\,m\,s^{-1}}$	mv
I_G, I_0	Mass moment of inertia	$\mathrm{kg\,m^2}$	$\int_{\underset{\text{mass}}{M}} r^2\,\mathrm{d}m$
M, τ	Moment/Torque	N m	Fd
k	Spring constant	$\mathrm{N\,m^{-1}}$	$\frac{F}{x}$

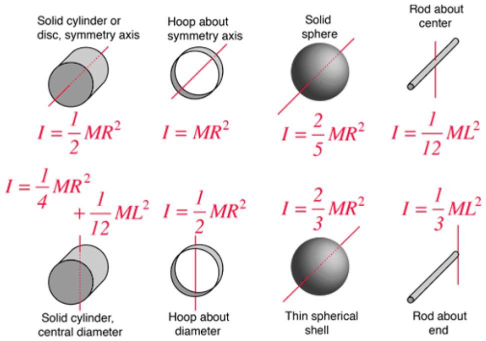


Figure 1: Mass moment of inertia