THE UNIVERSITY OF SYDNEY

SCHOOL OF MATHEMATICS

Math1062 Cheat Sheet

2025 Final

Calculus

Quick Facts

- General solution: all possible solutions in a DE
- Particular solution: solution evaluated through a point

Slope fields, show the range of solutions a DE can have.

Linear DEs

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$r(x) = \exp\left(\int p(x)dx\right)$$

$$\frac{d}{dx}\left(y\, r(x)\right) = q(x)r(x)$$

$$y = \frac{1}{r(x)}\left(\int q(x)r(x)dx + c\right)$$

2^nd order homogeneous DE

Standard form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Behaviour of the roots,

1. Real and *distinct*, $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2. Real and *repeated*, $y(x) = (Ax + B)e^{\lambda x}$
3. Complex and conjugate, $y(x) = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$ with $\lambda_{1,2} = \alpha \pm i\beta$

Inhomogeneous DEs

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$$

Solve like,

1. Solve homogeneous part, $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$
2. Find particular solution to inhomogeneous part, $q(x)$ transforming it in such a way to allow for simplification
3. Combine, $y(x) = \overbrace{y_h(x)U}^{\text{homogeneous}} + \overbrace{y_i(x)}^{\text{inhomogeneous}}$

Coupled 1^st order DE

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$
$$\Rightarrow \frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$$

Multivariable Calculus - Curves

Domain restriction, original domain is $f(x, y) : D \to \mathbb{R}$, with a domain of $\mathbb{R}^2$.

Then with restriction,

$\mathfrak{ex}$, $f(x, y) = \sqrt{x + 2y}$

Domain: $\{(x, y) \in \mathbb{R}, x + 2y \geq 0\}$

$\therefore x + 2y = 0$

$$\Rightarrow y = -\frac{1}{2}x$$

Partial derivatives

Tangent equation on a plane,

$$z - z_1 = f_x(a, b)(x - x_a) + f_y(a, b)(y - y_b)$$

Total derivatives, used in situations

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

With, $y = f(t)$, and $x = g(t)$

Approximating

With $f(a, b)$ given can approximate the function with, α being close to a and β being close to b,

$f(a,b) - f(\alpha,\beta) \approx dz$

$$\Rightarrow dz \approx f_x(\alpha,\beta)\overbrace{(\Delta\alpha)}^{dx} + f_x(\alpha,\beta)\overbrace{(\Delta\beta)}^{dy}$$

Implicit differentiation

When $f(x,y) = k$ and $k \in \mathbb{R}$,

$$\frac{dy}{dx} = \frac{-f_x(x,y)}{f_y(x,y)}$$

Directional differentiation

Finding cases of direction on a plane,

$$D_u f(a,b) = f_x(a,b)u_x + f_y(a,b)u_y$$

with u being a unit vector of a certain direction.

1. Steepest ascent, $\nabla f(x,y) = f_x\mathbf{i} + f_y\mathbf{j}$
2. Steepest *descent*, $-\nabla f(x,y)$
3. Zero-gradient, $\langle a,b\rangle \perp \nabla f(x,y)$

Classification of critical points

With $D = f_{xx}f_{yy} - (f_{xy})^2$ and $f_{xy} = f_{yx}$

$D(a,b)$	f_{xx}	Behaviour
< 0	-	Saddle
> 0	> 0	Local min
> 0	< 0	Local max
0	-	Inconclusive

Global extrema

Steps to find:

1. find critical points of the function, $f_x(a,b) = f_y(a,b) = 0$
2. find the values of the bounds, i.e., the corners
3. find the critical points on the edges, like for the edge $y = 1-x$, $x = t$, $y = 1-t$

Statistics

Statistics Terms

Sample standard deviation,

$$s = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

Population standard derivation,

$$\sigma = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

To convert to population standard in R [1] do,

```
popsd = sqrt((n-1)/n)*sd(data)
```

Hypothesis Testing

HATPC testing,

1. **Hypothesis:**
   (a) Null hypothesis (H_0), the tested condition, usually $\mu = c$, or $\mu_a - \mu_b = 0$
   (b) Alternative hypothesis (H_a), the negation of the hypothesis,
      i. Two-tailed tests, $\mu \neq c$
      ii. One-tailed tests, $\mu < c$
2. **Assumptions**, usually that the data is random and trails are independent and that the data is normal (follows CLT).
   (a) Can check normality with the QQ-plot and the boxplot, look for normality in those.
3. **Test-statistic**, done using the test-statistic
4. **P-value**, usually done to compare to the confidence interval
   (a) If CI is 95%, then if $p < 0.05$ H_0 is rejected otherwise it is retained.
5. **Conclusion**, then using the p value, and checking if $p > \alpha$ or $p < \alpha$
   (a) REJECT H_0
   (b) RETAIN H_0/FAIL TO REJECT H_0

[1] Whose `sd` command uses sample σ by default

Quick facts

- 95% confidence interval = 5% significance level (α)
- *Confidence interval* is from the observed value, i.e., $\bar{x}$ used to estimate population values Done with $[\bar{x} - \omega_{CI} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + \omega_{CI} \times \frac{\sigma}{\sqrt{n}}]$
- *Prediction interval* is from the 'given' value, i.e., p_0. Done with $[\mu - \omega_{CI} \times \frac{\sigma}{\sqrt{n}}, \mu + \omega_{CI} \times \frac{\sigma}{\sqrt{n}}]$
  with ω_{CI} being the value from `qnorm`/`qt`$(\frac{\alpha}{2} + (1-\alpha))$ `pnorm`: z-score $\rightarrow$ area; `qnorm`: area $\rightarrow$ z-score

Test statistics

Choosing a test,

Hypothesis Test for proportions

Use H_0 value,

$$E_0(\overline{X}) = \mu_0$$

$$SE_0(\overline{X}) = \sigma = \sqrt{\frac{\mu_0(1-\mu_0)}{n}}$$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

$$p = \texttt{pnorm}(z)$$

T-test

Used when the data has an unknown population standard deviation and the sample size is small ($< \sim 30$)

$$t = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

$$p = \texttt{pt}(z, \texttt{df} = n-1)$$

For two related series, μ_0 can be η and $\bar{x}$ can be η_2 in $d = \eta_2 - \eta = 0$.

Two sample T-tests

When there is two 'related' data sets gathered, and where the population parameters are not known.

$$H_0 : u_a - u_b = d = 0$$

$$H_a : d \neq 0$$

$$t = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\sigma_1^2/\overline{x}_1 + \sigma_2^2/\overline{x}_2}}$$

$$\text{df} = \frac{\left[\sigma_1^2/n_1 + \sigma_2^2/n_2\right]^2}{(\sigma_1^2/n_1)^2/(n_1-1) + (\sigma_2^2/n_2)^2/(n_2-1)}$$

Equal variance t-test

When the samples are not 'related' (i.e., samples are from different people, and sample sizes are not the same) but sample standard deviations are assumed to be roughly equal.

$$\hat{\sigma}_p = \sqrt{\frac{n \times \sigma_n^2 + m_0 \times \sigma_0^2}{n + m_0 - 2}}$$

$$t = \frac{\overline{x}_n - \overline{x}_0}{\hat{\sigma}_p \sqrt{1/m_0 + 1/n}}$$

Regression lines

$$SSE = \sum_{i=1}^{n} (y_1 - \overline{y})^2$$

$$\frac{SSE}{SST} = \frac{SD_e^2}{\sigma_y^2}$$

$$r^2 = 1 - \frac{SSE}{SST}$$

$$y = \beta_0 + \beta_1 x + \varepsilon$$

$$\text{df} = n - 2$$

$$\beta_1 = r\frac{\sigma_x}{\sigma_y}$$

$$\beta_0 = \overline{y} - b_1\overline{x}$$

As $r^2 \to 1$, the more *successful* the model is.

Inference for slopes

$H_0 : \beta_0 = 0$, so no correlation/relationship.

Assumptions

1. Linearity, scatter plot
2. Independence
3. Homoscedasticity, residual plot
4. Normally distributed errors, QQ-plot of residuals

$$t = \frac{\beta_1}{\sigma_{\beta_1}}$$

χ^2 test for independence

$$\text{expected}_{\#} = \frac{\sum_r \times \sum_c}{\sum_x \sum_y}$$

$$p = \texttt{pchisq}(t, \texttt{df} = (r-1)(c-1))$$